10. K. T. Compton and E. S. Lamar, "A test of the classical 'momentum transfer theory' of accomodation coefficients of ions at cathodes," Phys. Rev., 44, No. 9 (1933).
11. S. P. Perov, "Measurement of accomodation coefficient of air on metal filaments," Tr. Tsentr. Aerol. Obser., 61 (1965).

TWO-WAVE MODEL OF THE PROPAGATION OF PERTURBATIONS IN A LIQUID WITH GAS BUBBLES

V. G. Gasenko, V. E. Nakoryakov, and I. R. Shreiber

UDC 532.529

The speed of sound is dispersed in the propagation of the sound in a liquid containing gas bubbles. The form of the dispersion curve is well known [1]: for bubbles of one size, there are two dispersion branches divided by a region of opacity. Propagation of acoustic waves was examined in the Burgers–Korteveg–de Vries (BKV) approximation in [2-4]. This approximation corresponds to only one (the left) branch of the dispersion curve.

A model of perturbation propagation is proposed in the present work which corresponds to both branches simultaneously. An equation is derived and its transient solutions are found. Ranges of application of the low-frequency and high-frequency approximations of the equation are established.

1. In studying long-wave perturbations, either the effect of the compressibility of the liquid is ignored or it is reduced to contributing to dispersion of the wave through acoustic radiation with pulsations of the bubbles in the wave [2-6]. A more complete account of the compressibility of the liquid leads to the appearance of a second branch in the dispersion curve $c_f = c_f(\omega)$, where $c_f = \omega/k$ is the phase velocity; ω is the frequency of the superimposed perturbation; k is the wave number. The equation for the dispersion curve has the form

$$c_f^{-1} = \left(\frac{1}{c_{01}^2(1-m^2)} - \frac{m_1^2}{c_2(1-m_1^2)}\right)^{1/2}, \tag{1.1}$$

$m_1 = \omega/\omega_0$; $\omega_0^2 = 3\gamma p_{02}/R_0^2\rho_1$ is the square of the resonance frequency of the bubble; p_{02} is the pressure of the gas in the bubble; ρ_1 is the density of the liquid; R_0 is the initial radius of the bubble; γ is the adiabatic exponent; $c_{01}^2 = \gamma p_{02}/\rho_0\varphi_0$ is the square of the low-frequency approximation of the speed of sound in a liquid with gas bubbles; φ_0 the initial gas content; c_2 is the speed of sound in the liquid. The curve of Eq. (1.1) is shown in Fig. 1a. The initial section of the left side of the dispersion curve corresponds to the Korteveg–de Vries (KV) equation [2-4]

$$\partial p/\partial t + c_{01}\partial p/\partial x - \beta c_{01}\partial^3 p/\partial x^3 = 0$$

(p is the pressure in the mixture). The right branch of the dispersion curve corresponds to the Klein–Gordon equation

$$\frac{\partial^2 p}{\partial t^2} - c_2^2\frac{\partial^2 p}{\partial x^2} = -\frac{c_2^2}{2\beta}\delta p = -\frac{c_2^2\omega_0^2}{c_{01}^2}\delta p.$$

Shown below is the derivation of an equation corresponding to both branches of the dispersion curve (Fig. 1a).

2. Let us turn to equations describing the gadynamics of a homogeneous bubble suspension

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}; \quad (2.2) \quad \frac{\partial \rho}{\partial t} = -\rho\frac{\partial u}{\partial x} \tag{2.1}$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 119-127, November-December, 1979. Original article submitted November 2, 1978.

or, equivalent to system (2.1), (2.2), the single equation

$$\partial^2\rho/\partial t^2 = \partial^2 p/\partial x^2 \tag{2.3}$$

(ρ is the density of the mixture and u is the sound perturbation in the mixture). We will assume that most of the nonlinearity in the above problem is introduced by the equation of state, which is derived from the equation for the pulsation of a single bubble and from elementary relations of the homogeneous model in [1, 4]. If we ignore the dissipations, including dissipation due to acoustic radiation, that occur when compressibility is considered, we have the following equation of state:

$$\delta p = c^2\delta\rho + \frac{c^2}{\omega_1^2}\left(\frac{\partial^2\rho}{\partial t^2} - \frac{1}{c_2^2}\frac{\partial^2 p}{\partial t^2}\right). \tag{2.4}$$

Consideration is made of compressibility in Eq. (2.4) only in the changeover from R to ρ in the Rayleigh equation. In Eq. (2.4), c^2 is the square of the speed of sound, plotted from current characteristics of the wave:

$$c^{-2} = \rho\varphi/\gamma p + (1-\varphi)/c_2^2, \quad \text{for} \quad \varphi < 10^{-2} \quad c^{-2} \approx \rho\varphi/\gamma p.$$

At low amplitudes of the initial perturbation, c^2 may be represented through the initial parameters of the medium and the current amplitude of the wave

$$c^2 = c_{01}^2 + \frac{-2\varphi_0 + (\gamma+1)(1-c^2/c_2^2)}{\rho_1\varphi_0} c_{01}^2\delta\rho. \tag{2.5}$$

The parameter relation $c^2/\omega_1^2 = R^2/3\varphi(1-\varphi)$ is slightly dependent on wave amplitude and will henceforth be assumed constant

$$c^2/\omega_1^2 = R_0^2/3\varphi_0(1-\varphi_0) = 2\beta, \tag{2.6}$$

where β is the dispersion parameter in the BKV equation [2-4]. Taking (2.5), (2.6) into account, Eq. (2.4) may be written in the form

$$\delta p = c_{01}^2\delta\rho + 2c_{01}^2\frac{\Delta p}{p_0}\delta\rho + 2\beta\left(\frac{\partial^2\rho}{\partial t^2} - \frac{1}{c_2^2}\frac{\partial^2 p}{\partial t^2}\right). \tag{2.7}$$

Twice differentiating Eq. (2.7) over time and using the gasdynamics equation of the mixture in the form of (2.3), we obtain a single equation relative to the pressure perturbation

$$\frac{\partial^2 p}{\partial t^2} - c_{01}^2\frac{\partial^2 p}{\partial x^2} - 2\frac{d^2}{dx^2}c_{01}^2\frac{(\Delta p)^2}{p_0} = -\frac{2\beta}{c_2^2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 p}{\partial t^2} - c_2^2\frac{\partial^2 p}{\partial x^2}\right). \tag{2.8}$$

Substitution in the nonlinearity is done for the relation $\delta p = c_{01}^2\delta\rho$, valid to a first approximation. The nonlinearity $\frac{d^2}{dt^2}\left(\frac{\Delta p^2}{p_0}\right) = c_{01}^2\frac{d^2}{dx^2}\left(\frac{(\Delta p)^2}{p_0}\right)$. Linearized equation (2.8) corresponds to dispersion relation (1.1). For low frequencies $\omega < \omega_0$, Eq. (2.8) transforms into Boussinesq's equation

$$\frac{\partial^2 p}{\partial t^2} - c_0^2\frac{\partial^2 p}{\partial x^2} - 2c_{01}^2\frac{\partial^2}{\partial x^2}\left(\frac{(\Delta p)^2}{p_0}\right) - 2\beta\frac{\partial^4 p}{\partial t^2\partial x^2} = 0.$$

This equation contains as solutions waves traveling to the right and to the left, which makes it possible to use it as a basis for examining problems connected with reflection of a wave. For waves traveling in one direction, this equation transforms into the KV equation

$$\frac{\partial p}{\partial t} + c_{01}\frac{\partial p}{\partial x} + c_0\frac{\Delta p}{p_0}\frac{\partial p}{\partial x} - \beta c_{01}\frac{\partial^3 p}{\partial x^3} = 0.$$

For frequencies $\omega \geq \omega_0 c_{01}/c_2$, Eq. (2.8) transforms into the nonlinear Klein–Gordon equation

$$\frac{\partial^2 p}{\partial t^2} - c_2^2\frac{\partial^2 p}{\partial x^2} = -\frac{c_2^2}{2\beta}(\delta p - 2(\delta p)^2). \tag{2.9}$$

Wave propagation in a liquid with gas bubbles was studied in [7] in an approximation of the linear Klein–Gordon equation. Linearized equation (2.9) corresponds to the right branch of

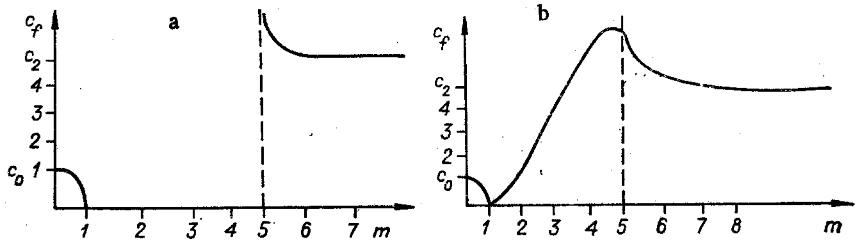

Fig. 1

the dispersion curve. In the general case, Eq. (2.8) corresponds to both branches of the dispersion curve (see Fig. 1a) and contains two characteristic speeds of sound c_{01} and c_2. Henceforth, this equation will be referred to as a two-wave nonlinear equation (TNE).

Accounting for dissipation and the nonlinearity of the oscillations of a single bubble leads to a more complete equation of state

$$\delta p = c^2\delta\rho + 2\mu_1\left(\frac{\partial\rho}{\partial t} - \frac{1}{c_2^2}\frac{\partial p}{\partial t}\right) + 2\beta\left(\frac{\partial^2\rho}{\partial t^2} - \frac{1}{c_2^2}\frac{\partial^2 p}{\partial t^2}\right) + 2k_1\left(\rho^2/c_{01}\partial\rho/\partial t - \rho^2/c_2^2\partial\rho/\partial t\right), \tag{2.10}$$

where μ_1 is the dissipation coefficient of the mixture, including viscous, acoustic, and thermal losses; $k_1 = (1/6)R_0^2/\varphi_0$ is the coefficient of the greatest nonlinearity.

Equation of state (2.10) leads to a more complete two-wave equation which more accurately describes empirical results

$$p_{tt} - c_{01}^2 p_{xx} - 2c_{01}^2\left(\frac{\Delta p^2}{p}\right)_{xx} = -\frac{2\beta}{c_2^2}\frac{d^2}{dt^2}\left(p_{tt} - c_2^2 p_{xx}\right) - \frac{2\mu_1}{c_2^2}\frac{d}{dt}\left(p_{tt} - c_2^2 p_{xx}\right) - \frac{2k_1'}{c_2^2}\frac{d^2}{\partial t^2}(\rho_0^2)\left(\frac{dp}{dt}\right)^2. \tag{2.11}$$

3. Let us examine the low- and high-frequency approximations of Eq. (2.11) in greater detail. We will look at the left branch of the dispersion curve, describing subresonance frequencies. To a first approximation for $(\Delta p/p_0) \approx \varepsilon$, this branch can be described by the wave equation

$$\frac{\partial^2 p}{\partial t^2} - c_{01}^2\frac{\partial^2 p}{\partial x^2} = o(\varepsilon^2). \tag{3.1}$$

In this case, Eq. (2.11) transforms into the Navier–Stokes–Boussinesq equation

$$\frac{\partial^2 p}{\partial t^2} - c_{01}^2\frac{\partial^2 p}{\partial x^2} - 2c_0^2\frac{\partial^2}{\partial x^2}\left(\frac{(\Delta p)^2}{p_0}\right) = -\frac{2\beta}{c_2^2}\frac{d^2}{dt^2}\left(1 - \frac{c_{01}^2}{c_2^2}\right)\frac{d^2 p}{dx^2} - \frac{2\mu_1}{c_2^2}\left(1 - \frac{c_{01}^2}{c_2^2}\right)\frac{\partial^3 p}{\partial x^2\partial t} - \frac{2k_1}{c_2^2}\frac{d^2}{dt^2}\left(1 - \frac{c_{01}^2}{c_2^2}\right)\rho_0^2\left(\frac{dp}{dt}\right)^2. \tag{3.2}$$

The transition to a unidirectional wave is made as follows. Wave operator (3.1) is represented in the form

$$\frac{\partial^2 p}{\partial t^2} - c_{01}^2\frac{\partial^2 p}{\partial x^2} = \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial t} + c_{01}\frac{\partial p}{\partial x}\right) - c_0\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial t} + c_0\frac{\partial p}{\partial x}\right). \tag{3.3}$$

For determinacy, we will choose a wave traveling to the right (the characteristic $x - c_0 t = \xi$). In this wave, the following relation is valid

$$\partial/\partial x = -1/c_{01}\cdot\partial/\partial t + o(\varepsilon^2). \tag{3.4}$$

Using Eqs. (3.3) and (3.4), Eq. (3.2) is integrated once over time to obtain the BKV equation

$$\frac{\partial p}{\partial t} + c_0\frac{\partial p}{\partial x} + c_{01}\frac{\Delta p}{p_0}\frac{\partial p}{\partial x} + \mu_1\left(1 - \frac{c_{01}^2}{c_2^2}\right)\frac{\partial^2 p}{\partial x^2} + \beta c_0\left(1 - \frac{c_{01}^2}{c_2^2}\right)\frac{\partial^3 p}{\partial x^3} + k_1 c_{01}\rho_0^2\left(1 - \frac{c_{01}^2}{c_2^2}\right)\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right)^2 = 0. \tag{3.5}$$

The last term in (3.5) may be written approximately in the form

$$k_1 c_{01}\rho^2\left(1 - \frac{c_{01}^2}{c_2^2}\right)\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right)^2 \approx k_1 c_{01}\rho_0^2\left(1 - \frac{c_{01}^2}{c_2^2}\right)\left(\frac{\partial p}{\partial x}\right)\frac{\partial^2 p}{\partial x^2}.$$

This nonlinear term may correspond to wave intensification and "sharpening" of the wave structures. Nonlinearity of this nature was considered in a study of stationary waves in a bubble mixture [8]. On the right branch, where $\omega > \omega_0 c_{01}/c_2$, the following is valid to a first approximation

$$\frac{\partial^2 p}{\partial t^2} - c_2^2 \frac{\partial^2 p}{\partial x^2} = o(\varepsilon^2)$$

and Eq. (2.11) transforms into the nonlinear Klein–Gordon equation with dissipation

$$2\beta\left(\frac{1}{c_2^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2}\right) + \left(1 - \frac{c_{01}^2}{c_1^2}\right)\delta p = \frac{2c_2}{c_{01}^2}\delta p^2 + 2\mu_1(c_2^2 - c_{01}^2)/c_2^2 c_{01}^2 \frac{\partial p}{\partial t} - 2k_1\rho_0^2(c_2^2 - c_{01}^2)/c_2^2 c_{01}^2\left(\frac{\partial p}{\partial t}\right)^2.$$

The dispersion curve corresponding to linearized equation (2.11) is shown in Fig. 1b. The presence of dissipation in the system "smears" the "opacity window." It should be noted that accounting for distribution with respect to dimensions R_0 at the initial moment of time in the mixture also leads to a continuous dispersion curve and the appearance of damping in the system. This effect is discussed in [3, 6, 9].

4. Numerical modeling of the propagation of perturbations in a liquid with gas bubbles was done on the basis of Eq. (2.8). With the following substitutions,

$$\tau = c_{01}t/\sqrt{\beta},\ \xi = x/\sqrt{\beta},\ u = 2\Delta p/p_0,\ V = c_2/c_{01},$$

Eq. (2.8) may be reduced to dimensionless form

$$u_{\tau\tau} - u_{\xi\xi} - (1/2)(u^2)_{\xi\xi} - V^{-2}(u_{\tau\tau} - u_{\xi\xi})_{\tau\tau} = 0. \quad (4.1)$$

Only one parameter V is introduced in Eq. (4.1), so that the nature of the solutions to (4.1) depends significantly on the type of initial perturbation.

In the general case, Eq. (4.1) describes a wave propagating to both sides. To separate waves propagating to one side only, we solved the following boundary-value problem: zero initial conditions were chosen, and the boundary condition on the left and of the integration interval [0, L] was assigned in the form

$$u(0, \tau) = \begin{cases} 0, & \tau < 0, \\ u_0 \exp[-(\tau/\tau_0)^2], & \tau \geqslant 0. \end{cases} \quad (4.2)$$

This corresponds to the appearance of a finite signal with a leading edge of limiting steepness. To determine the nature of the solutions of Eq. (4.1) relative to parameters V, u_0, and τ_0, we also solved the limiting cases of Eq. (4.1) corresponding to:

a) the low-frequency branch

$$u_{\tau\tau} - u_{\xi\xi} - (1/2)(u)_{\xi\xi}^2 - u_{\xi\xi\tau\tau} = 0; \quad (4.3)$$

b) the high-frequency branch

$$V^2 u_{\tau\tau} - u_{\xi\xi} + u(1 - V^{-2}) - 2V^{-2}u^2 = 0. \quad (4.4)$$

It should be noted that for gas contents $\varphi \approx 10^{-3}$–10^{-2} in gas–liquid mixtures, parameter $V^2 = 10$–10^2. Thus, the nonlinearity in Eq. (4.4) may be ignored, and we can use the linear Klein–Gordon equation on the high-frequency branch

$$V^2 u_{\tau\tau} - u_{\xi\xi} + u = 0. \quad (4.5)$$

Case a. Since Eq. (4.3) reduces to the KV equation for waves propagating to one side, it should be expected that the nature of the solutions of these equations will depend on the amplitude u_0 and duration τ_0 of the initial signal, i.e., will be determined by the similitude parameter $\sigma \approx \tau_0\sqrt{u_0}$ [10, 11] for nonlinear processes with dispersion. At $\sigma > \sigma_1$, where σ_1 is a certain critical value, solitary tones will predominate in the solution. At $\sigma < \sigma_1$, a wave packet will be formed.

Equation (4.3) is solved by means of numerical integration by a three-layer explicit–implicit difference diagram of second-order accuracy with respect to $\Delta\tau$ and $\Delta\xi$, and the solution fully supports this conclusion. Figure 2 shows the evolution of signal (4.2) (1) $\xi = 0$, 2) $\xi = 7.5$, 3) $\xi = 25$) at $\sigma = 5.6$, $u_0 = \sqrt{2}$, and $\tau_0 = 4$. It can be seen that in this case only solitons are formed. The value of σ_1, as the upper limit with respect to σ for the soliton solutions, lies within the range $11.2 > \sigma_1 > 5.6$. This coincides roughly with the value of σ for the KV equation, for which $\sigma_1 \approx 8$ [10]. It must be noted that the steepness of the leading edge of the initial signal has no effect on the result.

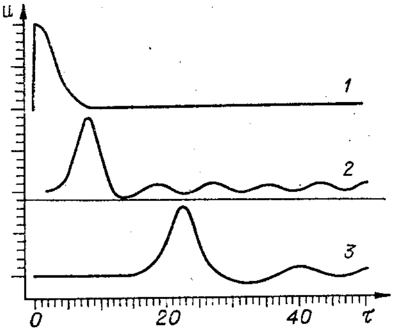

Fig. 2

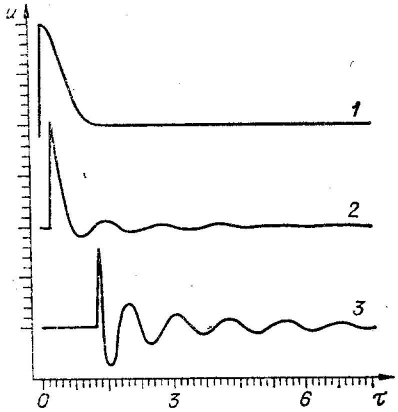

Fig. 3

Thus, evolution of a signal on the left branch practically repeats the well-known pattern found in studies based on the KV. The character of propagation of the waves may be predicted for any σ or u_0 and τ_0.

Case b. Evolution of a signal corresponding to the right branch of the dispersion curve was studied on the basis of Eq. (4.5). This equation is linear, and its solution depends not on u_0, but only on the duration of the signal τ_0. Comparison of the character of the solutions to Eq. (4.1), (4.3), and (4.5) has meaning only at identical values of τ_0, not σ, since the latter is a similitude parameter for the nonlinear process. Equation (4.5) was solved at V = 5 by means of numerical integration according to an explicit three-layer difference diagram with order of accuracy $o(\Delta\tau^2, \Delta\xi^2)$. Figure 3 shows the evolution of signal (4.2) at $\tau_0 = 1$. Curves 1-3 correspond to the time evolution of signal at points $\xi = 0$, 1, and 2.5. It can be seen that the solutions to Eqs. (4.3) and (4.5) are qualitatively different: Eq. (4.5) does not transform into the low-frequency part of the spectrum of the initial signal, so that all signals with a steep leading edge but of different durations evolve in qualitatively the same manner. It should be remembered that in case a) the steepness of the front has no effect on the evolution of the signal, while different durations of initial pulse lead to completely different patterns of signal evolution: either wave packets, or solitary tones [10, 12].

General Case c. Direct numerical integration of Eq. (4.2) according to a five-layer, implicit difference diagram with order of accuracy $o(\Delta\tau, \Delta\xi^2)$ showed that the character of the solution of this equation is not unambiguously determined by only one value of σ or only by τ_0. For a fairly large value, V = 5 in particular, evolution of a signal of type (4.2) with a steep leading edge has the following pattern at any values of σ and τ_0: a precursor traveling at the speed of sound in the liquid is generated in front of the main signal (the "main signal" here is taken to mean that part of the initial perturbation propagated at the speed of sound in the mixture, equal in the present case to unity). In the case in question, this speed is equal to 5. Qualitatively, the structure of this precursor coincides with the solution in Eq. (4.5). The evolution of the main, low-frequency component of the signal is determined, just as for Eq. (4.3), by the value of σ. At $\sigma < \sigma_1$ (here, σ_1 has almost the same value as for Eq. (4.3), i.e., $\sigma_1 \approx 8$), the main signal takes the form of a wave packet. At $\sigma > \sigma_1$, it takes the form of solitons.

The dynamics of the formation of the precursor for the case $u_0 = \sqrt{2}$, $\tau_0 = 1$ ($\sigma = 1.4 < \sigma_1$) are shown in Fig. 4, where curves 1-4 correspond to the time evolution of a signal of type (4.2) at the points $\xi = 0$, 1, 2.5, and 7.5. It should be noted that the damping of the precursor observed in this case is connected only with the numerical viscosity appearing on the right branch and due to the appearance of approximation term $\sim\Delta\tau u_{\tau\tau\tau}$, intrinsic to the chosen difference scheme, in the finite-difference analog of Eq. (4.1). Actually, the wave process being examined corresponds to the dispersion curve shown in Fig. 1b, which allows for the dissipation that takes place in actual wave propagation. A futher reduction in the value of σ due to signal duration leads to a reduction in the amplitude of the wave packet of the main signal, so that at $\tau_0 \sim 1/V(\omega = \omega c_2/c_{01})$, this amplitude may be ignored. In this case, evolution of the perturbation is described with a high degree of accuracy by Eq. (4.5). The critical value σ^*, beginning with which the signal behaves according to Eq. (4.5), is determined by the specific value of the initial amplitude u_0. For moderate amplitudes within

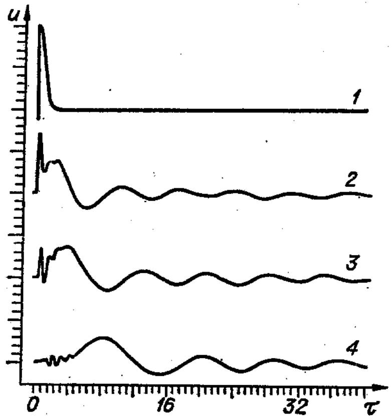

Fig. 4

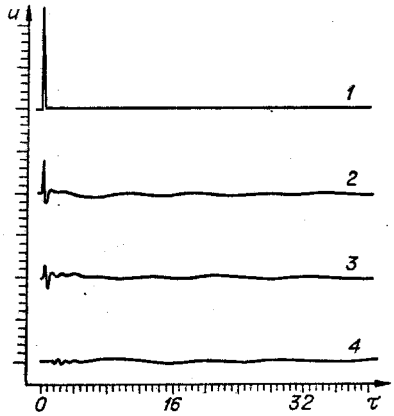

Fig. 5

the framework of the present approximation (u_0 = 1-10), $\sigma^* \ll \sigma_1$. Evolution of a signal of type (4.2) for Eq. (4.1) (at $u_0 = \sqrt{2}$, $\tau_0 = 0.05$, $\sigma = 0.07$), corresponding to this case, is shown in Fig. 5. Here, curves 1-4 correspond to the time evolution of the signal at points ξ = 0, 1, 2.5, and 7.5. All of the curves give the distribution of the signal over the time in fixed values of ξ, which correspond to the physical conditions of the experiment (a pressure transducer recording a signal over time was installed at a certain site). The frequency of the precursor $\omega = \omega_0 c_2/c_{01}$, while the frequency of the base signal $\omega \leq \omega_0$. It should be noted that a distinction can be made between the base signal and precursor by the above method only with respect to the time evolution. Division by wavelengths in the x-representation is not possible. Waves corresponding to the high-frequency branch also satisfy the long-wave requirements (the initial equations were written within the framework of these requirements). As before, the length of the waves of all oscillations exceeded the distance between bubbles and was, of course, much greater than the bubble dimensions. The approximation of a homogeneous medium was valid for all of the waves.

5. Let us examine the results of a numerical integration of Eqs. (4.1)-(4.5) and describe the evolution of perturbations in a liquid with gas bubbles, proceeding on the basis of the two-wave model. The two-wave equation that was derived describes well the evolution of perturbations with any steepness of initial signal and duration with respect to τ_0. For finite signals with build-up time and steepness greater than 1/V, the results of calculations by the TNE (4.1) and the Boussinesq equation (4.3) practically coincide. For finite signals of duration $\tau_0 \sim 1/V$, calculations by the TNE agree with the results of integration of the Klein-Gordon equation (4.5). For most of the perturbations realized in experiments, the steepness τ_0 of the leading edge is much less than unity and the duration of these pulses is considerably greater than unity. In this case, the correct pattern of signal evolution is given only by the TNE. Figure 4 shows the results of TNE calculations corresponding to this very case. The main part of the pulse evolves in accordance with the laws for low-frequency sound: it is transformed here into a wave packet, the period of oscillation of which is of the order of 1 and the velocity of which c_{01} = 1. The high-frequency part of the pulse is separated as a precursor — an alternating wave train — moving at the speed c_2 = 5. The period of oscillation of this train τV = 1.

At present, most of the empirical data available is from studies of the dynamics of perturbations in a liquid with gas bubbles corresponding to conditions for the left branch of the dispersion curve [12, 13]. However, the empirical results that have been obtained at the Institute of Thermophysics (Siberian Branch, Academy of Sciences of the USSR) [13] on the propagation of perturbations with steep leading edges indicates that, qualitatively speaking, there is agreement between the overall-pattern following from the two-wave model of precursor frequency in the experiment and the frequency following from the calculations. The precursor in this case moves at a speed c_2 and has a frequency $\omega = \omega_0 c_2/c_{01}$, plotted from the parameters of the mixture studied in the experiment in [13].

The first experimental observation of a precursor in a bubble mixture was in [6] in a study of shock waves initiated by an explosion. Shock waves with steep fronts of different duration were also studied in [7]. The attempt made in this work to describe all of these experiments on the basis of the Klein-Gordon approximation is unjustified. Most of these experiments can be described only on the basis of the TNE.

LITERATURE CITED

1. G. K. Batchelor, "Compression waves in a liquid suspension of gas bubbles," Sb. Per. Mekh., No. 3 (1968).
2. V. E. Nakoryakov, V. V. Sobolev, and I. R. Shreiber, "Long-wave perturbations in a gas—liquid mixture," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 5 (1972).
3. L. Von Weingarten, "Unidimensional flows of liquids with gas bubbles," in: Rheology of Suspensions [Russian translation], Mir, Moscow (1975).
4. V. E. Nakoryakov, V. V. Sobolev, and I. R. Shreiber, "Finite-amplitude waves in two-phase systems," in: S. S. Kutateladze (editor), Wave Processes in Two-Phase Media, Inst. Teplofiz. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1975).
5. R. I. Nigmatulin, A. I. Ivandaev, and A. A. Gubaidullin, "Numerical modelling of transient wave processes in dispersed two-phase media," in: Transactions of the Third International Seminar on Models of Continuum Mechanics [in Russian], Novosibirsk (1976).
6. V. K. Kedrinskii, "Propagation of perturbations in a liquid containing gas bubbles," Zh. Prikl. Mekh. Tekh. Fiz., No. 4 (1968).
7. N. V. Malykh and I. A. Ogorodnikov, "On the use of the Klein—Gordon equation to describe the structure of compression pulses in a liquid with gas bubbles," in: Continuum Dynamics [in Russian], Vol. 29, Inst. Teplofiz. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1977).
8. V. V. Goncharov, K. A. Naugol'nykh, and S. A. Rybak, "Stationary perturbations in a liquid containing gas bubbles," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1976).
9. D. D. Ryutov, "Landau damping analog in the problem of the propagation of a sound wave in a liquid with gas bubbles," Pisma Zh. Eksp. Teor. Fiz., 22, No. 9 (1975).
10. V. I. Karpman, Nonlinear Waves in Dispersive Media [in Russian], Nauka, Moscow (1973).
11. V. G. Gasenko, V. E. Nakoryakov, and I. R. Shreiber, "Burgers—Korteveg—de Vries approximation in the wave dynamics of gas—liquid systems," in: Nonlinear Wave Processes in Two-Phase Media [in Russian], Inst. Teplofiz. Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1977).
12. V. V. Kuznetsov, V. E. Nakoryakov, B. G. Pokusaev, and I. R. Shreiber, "Propagation of perturbations in a gas—liquid mixture," J. Fluid Mech., 85 (1978).
13. V. E. Nakoryakov, B. G. Pokusaev, I. R. Shreiber, V. V. Kuznetsov, and N. V. Malykh, "Experimental study of shock waves in a loquid with gas bubbles," in: Wave Processes in Two-Phase Media, op. cit.

INTERACTION OF SHOCK WAVES IN AN ELASTOPLASTIC MEDIUM WITH HARDENING

V. A. Baskakov

UDC 539.3

A study was made of the laws and the character of the deformation of an elastoplastic material after the passage of shock waves brought about by rather intense sources of perturbations. At a sufficiently great distance from the source, the fronts of the waves in the vicinity of the point of their interaction can be regarded as flat. The model of the medium provides for taking account of two hardening mechanisms [1]: kinematic and isotropic. Using the apparatus of the theory of fractures [2] and the method of [3-5], at first an elastic, and then an elastoplastic self-similar solution of the problem is constructed. The principal difficulty here consists in seeking the previously unknown lines separating the regions of elastic and plastic deformation of the material, at which the boundary conditions are assigned for the solution of a quasilinear system of differential equations in dissipative regions. A study is made of the effect of the hardening parameter on the qualitative side of the interaction of the waves. The basic relations were investigated using a digital computer; concrete numerical results were obtained. The solutions presented are a natural development of [5-7].

Let two flat shock waves in the form of steps Σ_1 and Σ_2 be propagated into an undeformed elastoplastic medium with the velocity G at an angle of $0 < 2\alpha < \pi$ (Fig. 1). Within the framework of the theory of small elastoplastic deformations it is postulated that the total

Voronezh. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 127-134, November-December, 1979. Original article submitted November 24, 1978.